NONISOMORPHIC COMPLETE SETS OF ORTHOGONAL F-SQUARES, HADAMARD MATRICES, AND DECOMPOSITIONS OF A 2^4 DESIGN

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ABSTRACT

The five nonisomorphic Hadamard matrices of order 16 given by Hall (1961) correspond to five distinct decompositions of a 2^4 design into single-degree-of-freedom orthogonal contrasts. The analysis of variance tables associated with these decompositions are compared, with emphasis on the three cases that have a 4×4 row and column structure. The three corresponding classes of Hadamard matrices generate nonisomorphic complete sets of nine orthogonal $F(4;2,2)$-squares, one of which shows a previously unreported pattern; the remaining two Hadamard classes do not produce complete sets of F-squares.

1. INTRODUCTION

A square matrix H of order n is a <u>Hadamard matrix</u> if each of its entries is +1 or -1 and each pair of its rows is orthogonal, that is, $HH' = nI$. For a general treatment of Hadamard matrices, see Hedayat and Wallis (1978). A Hadamard matrix is normalized if its first row and column consist entirely of +1's.

Two Hadamard matrices are isomorphic (or equivalent) if one can be obtained from the other by a sequence of row and column permutations and row and column complementations. Hall (1961) derived five nonisomorphic Hadamard matrices of order 16 and showed that any Hadamard matrix of order 16 must be isomorphic to one of these five. The set of all Hadamard matrices of order 16 therefore consists of five equivalence classes. The representatives H_1 to H_5 of these five classes given by Hall are reproduced in Table I. The symbols + and - represent the numbers +1 and -1, respectively, as they will throughout this paper. Thus, for example, the complement of a row of a Hadamard matrix is obtained by multiplying each element of the row by - .

The rest of this paper is organized as follows: Section 1 concludes with a brief discussion of Hadamard products, including two lemmas concerning rows of +'s and -'s, and F-squares. There is a natural one-to-one correspondence relating the equivalence classes of Hadamard matrices of order 16 and the decompositions of a 2^4 design into single-degree-of-freedom orthogonal contrasts. This correspondence is presented in Section 2. The three decompositions associated with ANOVA (analysis of variance) tables having a useful row and column structure to be defined in Section 2 are treated in detail. In Section 3, it is shown that the three corresponding Hadamard classes produce nonisomorphic complete sets of nine orthogonal $F(4;2,2)$-squares, one of which is not isomorphic to any previously produced set. Section 4 establishes that the remaining two Hadamard classes do not produce complete sets of F-squares. Section 5 contains concluding remarks.

The Hadamard product, element-by-element multiplication of equally-dimensioned vectors or matrices, will be denoted by $*$. For a general discussion of this operation, see Searle (1982, Sec. 2.8). The identity element for Hadamard multiplication of rows is the row $\underline{e} \equiv (+ + + \cdots +)$. A set of row vectors is closed under $*$

TABLE I

Nonisomorphic Hadamard Matrices of Order 16 Given by Hall

```
++++ ++++ ++++ ++++
++++ ++++ ---- ----
++++ ---- ++++ ----
++++ ---- ---- ++++
++-- ++-- ++-- ++--
++-- ++-- --++ --++
++-- --++ ++-- --++
++-- --++ --++ ++--
+-+- +-+- +-+- +-+-
+-+- +-+- -+-+ -+-+
+-+- -+-+ +-+- -+-+
+-+- -+-+ -+-+ +-+-
+--+ +--+ +--+ +--+
+--+ +--+ -++- -++-
+--+ -++- +--+ -++-
+--+ -++- -++- +--+
```

a. H_1

```
++++ ++++ ++++ ++++
++++ ++++ ---- ----
++++ ---- ++++ ----
++++ ---- ---- ++++
++-- ++-- ++-- ++--
++-- ++-- --++ --++
++-- --++ ++-- --++
++-- --++ --++ ++--
+-+- +-+- +-+- +-+-
+-+- +-+- -+-+ -+-+
+-+- -+-+ +-+- -+-+
+-+- -+-+ -+-+ +-+-
+--+ +--+ +--+ -++-
+--+ +--+ -++- +--+
+--+ -++- +--+ +--+
+--+ -++- -++- -++-
```

b. H_2

```
++++ ++++ ++++ ++++
++++ ++++ ---- ----
++++ ---- ++++ ----
++++ ---- ---- ++++
++-- ++-- ++-- ++--
++-- ++-- --++ --++
++-- --++ ++-- --++
++-- --++ --++ ++--
+-+- +-+- +-+- +-+-
+-+- +-+- -+-+ -+-+
+-+- -+-+ +--+ +--+
+-+- -+-+ -++- -++-
+--+ +--+ +--+ -++-
+--+ +--+ -++- +--+
+--+ -++- +-+- -+-+
+--+ -++- -+-+ +-+-
```

c. H_3

```
++++ ++++ ++++ ++++
++++ ++++ ---- ----
++++ ---- ++++ ----
++++ ---- ---- ++++
++-- ++-- ++-- ++--
++-- ++-- --++ --++
++-- --++ ++-- --++
++-- --++ --++ ++--
+-+- +-+- +-+- +-+-
+-+- +--+ +--+ -+-+
+-+- -++- -+-+ +--+
+-+- -+-+ -++- -++-
+--+ +-+- -+-+ -++-
+--+ +--+ -++- +--+
+--+ -++- +-+- -+-+
+--+ -+-+ +--+ +-+-
```

d. H_4

```
++++ ++++ ++++ ++++
++++ ++++ ---- ----
++++ ---- ++++ ----
++++ ---- ---- ++++
++-- ++-- ++-- ++--
++-- ++-- --++ --++
++-- --++ +-+- +-+-
++-- --++ -+-+ -+-+
+-+- +-+- ++-- --++
+-+- +-+- --++ ++--
+-+- -+-+ +--+ -++-
+-+- -+-+ -++- +--+
+--+ +--+ +-+- -+-+
+--+ +--+ -+-+ +-+-
+--+ -++- +--+ +--+
+--+ -++- -++- -++-
```

e. H_5

if the Hadamard product of any two vectors in the set is also contained in the set. A <u>closed triple</u> is a set S of three nonidentity rows such that $S \cup \{\underline{e}\}$ is closed. Closed triples are strongly related to the triples defined from block design considerations and tabled for H_1 to H_5 by Hall (1961). If he had used row numbers instead of column numbers to represent the block design objects, his triples would be identical to the closed triples treated here.

The following two lemmas apply to rows of any length. They will be useful in Sections 2 and 4. The first simplifies the process of finding closed triples of rows. The second indicates that there is a special relationship between the row $\underline{e}$ and closed triples.

Lemma 1.1. The set $\{\underline{r}_1, \underline{r}_2, \underline{r}_3\}$ of three distinct rows whose elements are +'s and -'s is a closed triple iff (if and only if) $\underline{r}_1 * \underline{r}_2 = \underline{r}_3$.

Proof. The Hadamard product of any pair of these rows equals the remaining row, for instance, $\underline{r}_1 * \underline{r}_3 = \underline{r}_1 * (\underline{r}_1 * \underline{r}_2) = \underline{e} * \underline{r}_2 = \underline{r}_2$. QED

Lemma 1.2. If three orthogonal rows whose elements are +'s and -'s form a closed triple $\{\underline{r}_1, \underline{r}_2, \underline{r}_3\}$, then each of these rows is orthogonal to $\underline{e}$, that is, each row consists of equal numbers of +'s and -'s.

Proof. Let n denote the row length. Let #(+++) denote the number of row positions where rows $\underline{r}_1$, $\underline{r}_2$, and $\underline{r}_3$ contain +, +, and +, respectively; let #(++-) denote the number of positions where $\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$ contain +, +, -; and so on. By the last lemma, $\underline{r}_1 * \underline{r}_2 = \underline{r}_3$, so

$$\#(++-) = \#(+-+) = \#(-++) = \#(---) = 0 ,$$

$$\#(+++) + \#(--+) + \#(-+-) + \#(+--) = n .$$

The orthogonality of $\underline{r}_1$ to $\underline{r}_2$, $\underline{r}_1$ to $\underline{r}_3$, and $\underline{r}_2$ to $\underline{r}_3$ yields

$$\#(+++) + \#(--+) - \#(-+-) - \#(+--) = 0 ,$$

$$\#(+++) - \#(--+) + \#(-+-) - \#(+--) = 0 ,$$

$$\#(+++) - \#(--+) - \#(-+-) + \#(+--) = 0 .$$

Solving the last four equations gives $\#(+++) = \#(--+) = \#(-+-) = \#(+--) = \frac{1}{4}n$. The number of +'s in $\underline{r}_1$ equals $\#(+++) + \#(+--) = \frac{1}{2}n$, and $\underline{r}_2$ and $\underline{r}_3$ are treated similarly. QED

An *F(2λ;λ,λ)-square* *with* *two* *treatments* *or* *symbols* or simply an *F-square* is a $2\lambda \times 2\lambda$ matrix in which every row and column contains each of the two symbols exactly λ times. Two of these F-squares are *orthogonal* if each possible ordered pair of symbols appears together λ^2 times when one of the F-squares is superimposed on the other. A set of s F-squares is *mutually* *orthogonal* if every pair of these s F-squares is orthogonal. Such a set is denoted by $OF(2\lambda;\lambda,\lambda;s)$, and is *complete* when $s = (2\lambda-1)^2$. These definitions suffice for the presentation in this paper. More general definitions and discussion of the practical application of F-square designs are found in Hedayat and Seiden (1970) and Hedayat, Raghavarao, and Seiden (1975).

2. HADAMARD CLASSES AND 1-D.F. ANOVA TABLES

The standard analysis of a 2^4 design involves a grand mean M, main effects A, B, C, D, two-factor interactions AB, AC, AD, BC, BD, CD, three-factor interactions ABC, ABD, ACD, BCD, and four-factor interaction ABCD. These 16 terms for a column vector

$$\underset{\sim}{g}_1 \equiv [2M, A, B, AB, C, AC, BC, ABC, D, AD, BD, ABD, CD, ACD, BCD, ABCD]' ,$$

where the first entry of $\underset{\sim}{g}_1$ is twice the grand mean M. All other entries will be referred to simply as effects.

Using standard notation, define the vector of the 16 treatment combination means or cell means

$$\underset{\sim}{v} \equiv [(abcd),(abc),(abd),(ab),(acd),(ac),(ad),(a),(bcd),(bc),(bd),(b),(cd),(c),(d),(1)]' ,$$

where (abcd) denotes the mean of all observations receiving the treatment combination abcd, and so on. Then $\underset{\sim}{g}_1$ can be obtained from $\underset{\sim}{v}$ by

$$\underset{\sim}{g}_1 = \frac{1}{8} H_1 \underset{\sim}{v} ,$$

where the Hadamard matrix H_1 appears in Table Ia. The orthogonality of H_1 establishes easily the well-known fact that the last 15 elements of $\underline{g}_1$ form a set of mutually orthogonal single-degree-of-freedom contrasts. These contrasts partition the degrees of freedom in a 2^4 design.

The 16 rows of H_1 form a closed set. Table IIa gives the number of the row obtained as the Hadamard product of each pair of nonidentity rows of H_1, for example, row $2 * $ row $3 =$ row 4. Table IIIa lists the 35 closed triples, which are easily checked with the help of Table IIa and Lemma 1.1; five disjoint closed triples are starred. (Other choices of the five are possible, giving different but isomorphic results. The selection made here will facilitate the comparison of ANOVA tables related to H_1, H_2, and H_3.) Replace the row numbers in these triples by the corresponding entries of $\underline{g}_1$: for example, 2,9,10 by A,D,AD; 3,5,7 by B,C,BC; and so on. The result is five sets of three effects. Each set has the important property: (P) the interaction of any two of its **members is the set's remaining member, i.e., it is closed under** multiplication.

This division of the 15 effects of $\underline{g}_1$ into these five members is the set's remaining member.

This division of the 15 nonmean effects of $\underline{g}_1$ into these five sets of three effects produces the ANOVA table shown in Table IVa. It is related to the Factorial Complete Confounding Construction of Federer _et al_. (1969), which has been used to construct OL(4,3), the complete set of three orthogonal latin squares of order 4. Take a 2^4 factorial experiment with factors a, b, c, and d and write the treatment combinations in a 4×4 square whose rows are confounded with main effects A, D, and their interaction AD and whose columns are confounded with main effects B, C, and their interaction BC. (This choice of effects is again for convenience, agreeing with two of the starred triples.) The result, with 0 and 1 representing the two levels of each factor, is

TABLE II

Hadamard Products of Nonidentity Rows of H_1 to H_5

a. H_1

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	6	7	8	1	2	3	4	13	14	15	16	9	10	11	12
6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11
7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7
11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
12	11	10	9	16	15	14	13	4	3	2	1	8	7	6	5
13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
14	13	16	15	10	9	12	11	6	5	8	7	2	1	4	3
15	16	13	14	11	12	9	10	7	8	5	6	3	4	1	2
16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

b. H_2

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	6	7	8	1	2	3	4	0	0	0	0	0	0	0	0
6	5	8	7	2	1	4	3	0	0	0	0	0	0	0	0
7	8	5	6	3	4	1	2	0	0	0	0	0	0	0	0
8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0
9	10	11	12	0	0	0	0	1	2	3	4	0	0	0	0
10	9	12	11	0	0	0	0	2	1	4	3	0	0	0	0
11	12	9	10	0	0	0	0	3	4	1	2	0	0	0	0
12	11	10	9	0	0	0	0	4	3	2	1	0	0	0	0
13	14	15	16	0	0	0	0	0	0	0	0	1	2	3	4
14	13	16	15	0	0	0	0	0	0	0	0	2	1	4	3
15	16	13	14	0	0	0	0	0	0	0	0	3	4	1	2
16	15	14	13	0	0	0	0	0	0	0	0	4	3	2	1

TABLE II (cont.)

c. H_3

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	0	0	0	0	0	0	0	0
4	3	2	1	8	7	6	5	0	0	0	0	0	0	0	0
5	6	7	8	1	2	3	4	0	0	0	0	0	0	0	0
6	5	8	7	2	1	4	3	0	0	0	0	0	0	0	0
7	8	5	6	3	4	1	2	0	0	0	0	0	0	0	0
8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0
9	10	0	0	0	0	0	0	1	2	0	0	0	0	0	0
10	9	0	0	0	0	0	0	2	1	0	0	0	0	0	0
11	12	0	0	0	0	0	0	0	0	1	2	0	0	0	0
12	11	0	0	0	0	0	0	0	0	2	1	0	0	0	0
13	14	0	0	0	0	0	0	0	0	0	0	1	2	0	0
14	13	0	0	0	0	0	0	0	0	0	0	2	1	0	0
15	16	0	0	0	0	0	0	0	0	0	0	0	0	1	2
16	15	0	0	0	0	0	0	0	0	0	0	0	0	2	1

d. H_4

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	0	0	0	0	0	0	0	0
3	4	1	2	7	8	5	6	0	0	0	0	0	0	0	0
4	3	2	1	8	7	6	5	0	0	0	0	0	0	0	0
5	6	7	8	1	2	3	4	0	0	0	0	0	0	0	0
6	5	8	7	2	1	4	3	0	0	0	0	0	0	0	0
7	8	5	6	3	4	1	2	0	0	0	0	0	0	0	0
8	7	6	5	4	3	2	1	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

TABLE II (cont.)

e. H_5

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	0	0	0	0	0	0	0	0	0	0	0	0
4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0
5	6	0	0	1	2	0	0	0	0	0	0	0	0	0	0
6	5	0	0	2	1	0	0	0	0	0	0	0	0	0	0
7	8	0	0	0	0	1	2	0	0	0	0	0	0	0	0
8	7	0	0	0	0	2	1	0	0	0	0	0	0	0	0
9	10	0	0	0	0	0	0	1	2	0	0	0	0	0	0
10	9	0	0	0	0	0	0	2	1	0	0	0	0	0	0
11	12	0	0	0	0	0	0	0	0	1	2	0	0	0	0
12	11	0	0	0	0	0	0	0	0	2	1	0	0	0	0
13	14	0	0	0	0	0	0	0	0	0	0	1	2	0	0
14	13	0	0	0	0	0	0	0	0	0	0	2	1	0	0
15	16	0	0	0	0	0	0	0	0	0	0	0	0	1	2
16	15	0	0	0	0	0	0	0	0	0	0	0	0	2	1

TABLE III

Closed Triples of Rows of H_1 to H_5

2,3,4	3,14,16	6,10,13	2,3,4	3,14,16	2,3,4	2,3,4	2,3,4
2,5,6	4,5,8	6,11,16	2,5,6	4,5,8	2,5,6	2,5,6	2,5,6
2,7,8	4,6,7	*6,12,15	2,7,8	4,6,7	2,7,8	2,7,8	2,7,8
*2,9,10	4,9,12	7,9,15	*2,9,10	4,9,12	2,9,10	3,5,7	2,9,10
2,11,12	4,10,11	7,10,16	2,11,12	4,10,11	2,11,12	3,6,8	2,11,12
2,13,14	*4,13,16	7,11,13	2,13,14	*4,13,16	2,13,14	4,5,8	2,13,14
2,15,16	4,14,15	7,12,14	2,15,16	4,14,15	2,15,16	4,6,7	2,15,16
*3,5,7	5,9,13	8,9,16	*3,5,7		3,5,7		
3,6,8	5,10,14	8,10,15	3,6,8		3,6,8		
3,9,11	5,11,15	*8,11,14	3,9,11		4,5,8		
3,10,12	5,12,16	8,12,13	3,10,12		4,6,7		
3,13,15	6,9,14		3,13,15				
	a. H_1		b. H_2		c. H_3	d. H_4	e. H_5

TABLE IV

ANOVA Tables Obtained from H_1 to H_3

Source of Variation	Degrees of Freedom	
a. H_1 ANOVA Table		
Mean	1	
Rows	3	
A		1
D		1
AD		1
Columns	3	
B		1
C		1
BC		1
Latin square 1 treatments	3	
AB		1
CD		1
ABCD		1
Latin square 2 treatments	3	
AC		1
ABD		1
BCD		1
Latin square 3 treatments	3	
ABC		1
BD		1
ACD		1
Total	16	
b. H_2 ANOVA Table		
Mean	1	
Rows	3	
A		1
D		1
AD		1
Columns	3	
B		1
C		1
BC		1
Cyclic latin square treatments	3	
AB		1
G_{21}		1
G_{24}		1
AC	1	
ABC	1	
BD	1	
ABD	1	
G_{22}	1	
G_{23}	1	
Total	16	

TABLE IV (cont.)

c. H_3 ANOVA Table

Source of Variation	Degrees of Freedom	
Mean	1	
Rows	3	
A		1
D		1
AD		1
Columns	3	
B		1
C		1
BC		1
AB	1	
AC	1	
ABC	1	
G_{31}	1	
G_{32}	1	
G_{33}	1	
G_{34}	1	
G_{35}	1	
G_{36}	1	
Total	16	

$$
\begin{array}{cc}
 & \begin{array}{cccc} (B)_0,(C)_0 & (B)_1,(C)_0 & (B)_0,(C)_1 & (B)_1,(C)_1 \\ (BC)_0 & (BC)_1 & (BC)_1 & (BC)_0 \end{array} \\
\begin{array}{l} (A)_0,(D)_0,(AD)_0 \\ (A)_1,(D)_0,(AD)_1 \\ (A)_0,(D)_1,(AD)_1 \\ (A)_1,(D)_1,(AD)_0 \end{array} & \left[\begin{array}{cccc} 1 & b & c & bc \\ a & ab & ac & abc \\ d & bd & cd & bcd \\ ad & abd & acd & abcd \end{array}\right]
\end{array} . \quad (2.1)
$$

The nine interaction effects of $\underset{\sim}{g}_1$ that are not confounded with rows and columns form three more triples with property (P): AB,CD,ABCD; AC,ABD,BCD; and ABC,BD,ACD. Confounding symbols I to IV with each of these triples in turn as rows were confounded with A,D,AD, for example,

$$\begin{aligned}
\text{I:}\quad & (AB)_0, (CD)_0, (ABCD)_0 \\
\text{II:}\quad & (AB)_0, (CD)_1, (ABCD)_1 \\
\text{III:}\quad & (AB)_1, (CD)_0, (ABCD)_1 \\
\text{IV:}\quad & (AB)_1, (CD)_1, (ABCD)_0 \quad ,
\end{aligned}$$

and inserting these symbols into the appropriate cells of (2.1) produces three orthogonal latin squares.

The five sets of three effects in Table IVa thus can be associated with rows, columns, and three orthogonal latin squares. A 16-degree-of-freedom ANOVA table has <u>row and column structure</u> if it contains a grand mean effect and at least two disjoint sets of three has <u>row and column structure</u> if its rows include $\underline{e}$ and at least two disjoint closed triples. This structure will have an important role in the treatment of complete sets of F-squares in Sections 3 and 4.
treatment of complete sets of F-squares in Sections 3 and 4.

The Hadamard matrix H_1 transforms $\underline{v}$ into the vector $\underline{g}_1$, whose elements are effects associated with one-degree-of-freedom rows in the standard ANOVA table. Any Hadamard matrix isomorphic to H_1 produces the same ANOVA table except for permutation of the symbols A, B, C, D. The use of Hadamard matrices not isomorphic to H_1 will now be examined.

Using H_2 from Table I, define the vector

$$\underline{g}_2 \equiv \frac{1}{8} H_2 \underline{v} \quad .$$

As with H_1, the orthogonality of H_2 establishes that the last 15 elements of $\underline{g}_2$ are mutually orthogonal single-degree-of-freedom contrasts. The first twelve rows of H_1 and of H_2 agree, so

$$\underline{g}_2 = \frac{1}{8}[2M, A, B, AB, C, AC, BC, ABC, D, AD, BD, ABD, G_{21}, G_{22}, G_{23}, G_{24}]' \quad ,$$

where matrix algebra shows that

$$\begin{aligned}
G_{21} &= \tfrac{1}{2}(\ CD + ACD + BCD - ABCD) \\
G_{22} &= \tfrac{1}{2}(\ CD + ACD - BCD + ABCD) \\
G_{23} &= \tfrac{1}{2}(\ CD - ACD + BCD + ABCD) \\
G_{24} &= \tfrac{1}{2}(-CD + ACD + BCD + ABCD) \quad .
\end{aligned}$$

The rows of H_2 do not form a closed set. Table IIb gives the row number of Hadamard products of rows of H_2, with 0 indicating that this product is not a row of H_2, as is the case for row $5 *$ row 9. There are 19 closed triples, which are listed in Table IIIb; they are easily found by using Lemma 1.1. Three disjoint closed triples are starred, and no larger set of disjoint closed triples can be found. (As with H_1, other choices are possible.) Replacing the row numbers in these triples by the corresponding entries of $\underline{g}_2$ produces the three sets of three effects A, D, AD; B, C, BC; and AB, G_{21}, G_{24}. Each set has property (P), and no further sets with this property can be formed from the remaining six effects.

This division of the fifteen effects of $\underline{g}_2$ results in the ANOVA table shown in Table IVb, which has row and column structure. An isomorphic version of this table, with A, B, C, D permuted, was shown by Mandeli (1975) to correspond to a cyclic 4×4 latin square design. The three sets satisfying property (P) can be associated with rows, columns, and a cyclic latin square.

The same approach can be applied to

$$\underline{g}_3 \equiv \frac{1}{8} H_3 \underline{v} \quad .$$

The first 10 rows of H_1 and H_3 agree, so

$$\underline{g}_3 = \frac{1}{8}[2M, A, B, AB, C, AC, BC, ABC, D, AD, G_{31}, G_{32}, \cdots, G_{36}]' \quad ,$$

where matrix algebra gives

$$\begin{aligned}
G_{31} &= \tfrac{1}{2}(\ BD + ABD + CD - ACD) \\
G_{32} &= \tfrac{1}{2}(\ BD + ABD - CD + ACD) \\
G_{33} &= \tfrac{1}{2}(\ CD + ACD + BCD - ABCD) \\
G_{34} &= \tfrac{1}{2}(\ CD + ACD - BCD + ABCD) \\
G_{35} &= \tfrac{1}{2}(\ BD - ABD + BCD + ABCD) \\
G_{36} &= \tfrac{1}{2}(-BD + ABD + BCD + ABCD) \quad .
\end{aligned}$$

The rows of H_3 do not form a closed set. Table IIc gives the Hadamard products of rows of H_3, with 0 indicating a product that is not a row of H_3. Table IIIc lists the 11 closed triples, of which no more than two are disjoint. Replacing the row numbers in the starred triples by the corresponding entries of $\underline{g}_3$ produces the two sets of three effects A,D,AD; and B,C,BC . Each of these sets has property (P), and no further sets of this type can be formed.

This division of the fifteen effects of $\underline{g}_3$ results in the ANOVA table shown in Table IVc, which has row and column structure. The two sets satisfying property (P) can be associated with rows and columns.

Similar single-degree-of-freedom ANOVA tables can be derived for the vectors

$$\underline{g}_4 \equiv \frac{1}{8}H_4\underline{v}, \qquad \underline{g}_5 \equiv \frac{1}{8}H_5\underline{v} \ .$$

Hadamard products of rows and closed triples appear in Tables IId and IIId for H_4 and in Tables IIe and IIIe for H_5 . In neither case can two disjoint pairs of closed triples be found. A single closed triple corresponds to a set of three effects that can be associated with either rows or columns, but not both. Thus the ANOVA tables for these cases do not have row and column structure. This lack has important implications for the F-square methods treated in the next two sections. These ANOVA tables are straightforward to derive and will be omitted here.

The discussion of this section is summarized as follow. The nonisomorphic Hadamard matrices H_1 to H_5 correspond to five decompositions of a 2^4 design into a single-degree-of-freedom ANOVA table. Three of these are given in Table IV.

3. HADAMARD CLASSES GENERATING COMPLETE SETS OF F-SQUARES

Assume that a Hadamard matrix H of order 16 has row and column structure. Set aside the row $\underline{e}$ and two disjoint closed triples of rows, which may be associated with row and column structure in a 4×4 design. If the elements of each of the remaining nine rows of H are suitably rearranged and used to fill a 4×4 matrix, a complete set of nine orthogonal F(4;2,2)-squares is produced. When row and column structure is absent, however, the rows of H do not yield a complete set of orthogonal F-squares. These results will be established and applied in this section and the next.

For an effect E in g_1, let $\underset{-}{r}_E$ denote the row of H_1 for which $E = \frac{1}{8}\underset{-}{r}_E \underset{-}{v}$. For example, $\underset{-}{r}_A$, $\underset{-}{r}_B$, $\underset{-}{r}_C$, and $\underset{-}{r}_D$ denote rows 2, 3, 5, and 9 of H_1, respectively; $\underset{-}{r}_{AB}$ and $\underset{-}{r}_{CD}$ denote rows 4 and 13; $\underset{-}{r}_{BC}$ and $\underset{-}{r}_{AD}$ denote rows 7 and 10. Row 1 of H_1 is the identity $\underset{-}{e}$. Define the disjoint closed triples $R \equiv \{\underset{-}{r}_A, \underset{-}{r}_B, \underset{-}{r}_{AB}\}$ and $Q \equiv \{\underset{-}{r}_C, \underset{-}{r}_D, \underset{-}{r}_{CD}\}$. For any row vector $\underset{-}{x} = (x_1\ x_2\ \cdots\ x_{16})$, define the 4×4 matrix $\mathscr{J}(\underset{-}{x})$ whose (i,j)th entry is $x_{4(i-1)+j}$:

$$\mathscr{J}(\underset{-}{x}) \equiv \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} \end{bmatrix} .$$

Lemma 3.1. Let a Hadamard matrix H of order 16 have $\underset{-}{e}$, R, and Q among its rows. Then for any other row $\underset{-}{r}$ of H, $\mathscr{J}(\underset{-}{r})$ is an F(4;2,2)-square. When $\mathscr{J}(\underset{-}{r})$ is formed for each of the nine remaining rows of H, a complete set of mutually orthogonal F-squares is obtained.

Proof. The rows of $\underset{-}{e}$, R, and Q are

$$\begin{bmatrix} \underset{-}{e} \\ \underset{-}{r}_A \\ \underset{-}{r}_B \\ \underset{-}{r}_{AB} \\ \underset{-}{r}_C \\ \underset{-}{r}_D \\ \underset{-}{r}_{CD} \end{bmatrix} = \begin{bmatrix} ++++ & ++++ & ++++ & ++++ \\ ++++ & ++++ & ---- & ---- \\ ++++ & ---- & ++++ & ---- \\ ++++ & ---- & ---- & ++++ \\ ++-- & ++-- & ++-- & ++-- \\ +-+- & +-+- & +-+- & +-+- \\ +--+ & +--+ & +--+ & +--+ \end{bmatrix} . \qquad (3.1)$$

For any other row $\underset{-}{r}$ of H, let n_1, n_2, n_3, n_4 be the number of +'s among its first, second, third, fourth set of four elements. Let m_1 be the number of +'s among elements 1, 5, 9, and 13 of $\underset{-}{r}$; m_2 the number of +'s among elements 2,6,10,14; m_3 the number of +'s among elements 3,7,11,15; and m_4 the number among elements 4,8,12,16. Orthogonality of $\underset{-}{r}$ to the rows of $\underset{-}{e}$ and R yields

$$\begin{aligned}
\underline{e} &: n_1 + n_2 + n_3 + n_4 = 8\ , \\
\underline{r}_A &: n_1 + n_2 + 4 - n_3 + 4 - n_4 = 8\ , \\
\underline{r}_B &: n_1 + 4 - n_2 + n_3 + 4 - n_4 = 8\ , \\
\underline{r}_{AB} &: n_1 + 4 - n_2 + 4 - n_3 + n_4 = 8\ .
\end{aligned}$$

Thus $n_1 = n_2 = n_3 = n_4 = 2$, so each row of $\mathscr{I}(\underline{r})$ has 2 +'s and 2 -'s. Orthogonality of $\underline{r}$ to $\underline{e}$, $\underline{r}_C$, $\underline{r}_D$, and $\underline{r}_{CD}$ gives the same equations with n_1, n_2, n_3, n_4 replaced by m_1, m_2, m_3, m_4, so each column of $\mathscr{I}(\underline{r})$ has 2 +'s and 2 -'s, and $\mathscr{I}(\underline{r})$ is an F-square. The F-squares $\mathscr{I}(\underline{r})$ formed from any two remaining rows of H are orthogonal, since the Hadamard product of these F-squares contains equal numbers of +'s and -'s . The set of nine F-squares of this type is complete, because $9 = (4-1)^2$. QED

Lemma 3.2. If a Hadamard matrix H of order 16 has $\underline{e}$ and at least two disjoint closed triples S and T among its rows, then H is isomorphic to a Hadamard matrix whose first seven rows are $\underline{e}$, R, and Q . Also, H is isomorphic to H_1, H_2, or H_3 .

Proof. Permute the rows of H so $\underline{e}$ is the first row, followed by the three rows of S, then the three rows of T, then the remaining nine rows. Next, permute the columns to make the second row equal to $\underline{r}_A$. Follow this by permuting the first eight columns, and then permuting the second eight columns, to make the third row equal to $\underline{r}_B$. By closure of the triple, the fourth row must equal $\underline{r}_{AB}$. Call this matrix H_a .

Let n_1, n_2, n_3, n_4 be the number of +'s among the first, second, third, and fourth set of four elements of the fifth row of H_a . It was shown in the last lemma that $n_1 = n_2 = n_3 = n_4 = 2$, allowing the permutation of columns 1 to 4 of H_a, then columns 5 to 8, then 9 to 12, then 13 to 16, to make the fifth row equal to $\underline{r}_C$. Call this matrix H_b . Let $m_1, \cdots, m_8$ be the number of +'s in the first, $\cdots$, eighth pair of elements in the sixth row of H_b . The sixth row is orthogonal to $\underline{e}$ and R, so the equations in the proof of the last lemma give

$$m_1 + m_2 = m_3 + m_4 = m_5 + m_6 = m_7 + m_8 = 2 \quad .$$

Similarly, since the seventh row of H_b is the product of $\underline{r}_C$ and the sixth row by closure,

$$m_1 + 2 - m_2 = m_3 + 2 - m_4 = m_5 + 2 - m_6 = m_7 + 2 - m_8 = 2 \ .$$

Solving these equations shows that $m_1 = m_2 = \cdots = m_8 = 1$, so the columns of H_b can be permuted, a pair at a time, to make the sixth row equal to $\underline{r}_D$. When this has been done, the seventh row must equal $\underline{r}_{CD}$ by closure. Call the resulting matrix H_c. Complement any row that begins with - (if H was normalized, none will occur) and call the resulting matrix H_d. Both H_c and H_d are isomorphic to the original H, and both have $\underline{e}$, R, and Q as their first seven rows.

Hall (1961) noted that any Hadamard matrix of order 16 whose first row and column consist entirely of +'s is equivalent to one of the five matrices H_1 to H_5 under permutation of rows and columns. (This property does not hold for higher orders, where complementation broadens the equivalence classes.) The number of disjoint closed triples is invariant under row and column permutations. Therefore H_d, which has at least two disjoint closed triples, cannot be equivalent to either H_4 or H_5, each of which does not, so H_d must be equivalent to H_1, H_2, or H_3. QED

Theorem 3.1. Let H be any Hadamard matrix of order 16 with row and column structure. A complete set of orthogonal F-squares can be generated from H by performing a series of column permutations on H and then applying $\mathscr{J}$ to the nine rows not involved in establishing the row and column structure of H.

Proof. Transform H to H_c as in the proof of Lemma 3.2, and obtain the set of F-squares by applying $\mathscr{J}$ to each of the last nine rows of H_c as in Lemma 3.1. QED

This method will now be used on H_1, H_2, and H_3. Choose closed triples $S = \{\underline{r}_A, \underline{r}_D, \underline{r}_{AD}\}$ and $T = \{\underline{r}_B, \underline{r}_C, \underline{r}_{BC}\}$, which occur in H_1, H_2, and H_3. Transform the rows $\underline{e}, \underline{r}_A, \underline{r}_D, \underline{r}_{AD}, \underline{r}_B, \underline{r}_C, \underline{r}_{BC}$, which are rows 1,2,9,10,3,5,7, into the pattern of (3.1) by permuting the columns, writing them in the order 1,3,5,7,2,4,6,8,9,11,13,15,10,12,14,16.

Applying $\mathscr{J}$ to the remaining rows 4, 6, 8, and 11 to 16 of H_1, H_2, or H_3 gives a complete set of orthogonal F-squares. These three sets of F-squares are shown in Table V.

Definition. Two sets of F-squares are isomorphic if the F-squares of one set can be transformed into the F-squares of the other set by a series of row and column permutations on each F-square.

The set of F-squares obtained from H_1 is isomorphic to a complete set of orthogonal $F(4;2,2)$-squares given by Hedayat, Raghavarao, and Seiden (1975). (A minor error appears in their F-square F_3, whose last two rows must be interchanged.) The set of F-squares obtained from H_2 is isomorphic to a complete set given by Mandeli (1975), and to another complete set given by Schwager, Federer, and Mandeli (1982). However, the sets of F-squares obtained from H_1, H_2, and H_3 are not isomorphic to each other; the last of these is not isomorphic to any complete set reported previously.

Theorem 3.2. The three complete sets of orthogonal F-squares in Table V are nonisomorphic, that is, no two of these sets are isomorphic. Any complete set is isomorphic to one of these three.

Proof. The number of distinct row patterns occurring in an F-square is invariant under row and column permutations of the F-square. Consequently, row and column permutations on each F-square in a set cannot change the number of F-squares in the set that have a specified number of row patterns. Each of the nine F-squares derived from H_1 contains exactly two distinct row patterns. Two distinct row patterns occur in the first five F-squares derived from H_2, while the last four of the H_2 F-squares have four row patterns. Two row patterns appear in the first three F-squares derived from H_3, while the last six of the H_3 F-squares have four row patterns. The invariance of the number of two-row and four-row F-squares in a set implies that none of the three sets from Table V can be transformed into another of these sets by row and column permutations. Lemmas 3.1 and 3.2 prove the last part. QED

TABLE V

Complete Sets of Orthogonal F-Squares Obtained from H_1 to H_3

	++--	+-+-	+--+	++--	++--	+-+-	+-+-	+--+	+--+
	++--	+-+-	+--+	--++	--++	-+-+	-+-+	-++-	-++-
a. H_1:	--++	-+-+	-++-	++--	--++	+-+-	-+-+	+--+	-++-
	--++	-+-+	-++-	--++	++--	-+-+	+-+-	-++-	+--+
	‖	‖	‖	‖	‖				
	++--	+-+-	+--+	++--	++--	+-+-	+-+-	+--+	+--+
	++--	+-+-	+--+	--++	--++	-+-+	-+-+	-++-	-++-
b. H_2:	--++	-+-+	-++-	++--	--++	+--+	-++-	+-+-	-+-+
	--++	-+-+	-++-	--++	++--	-++-	+--+	-+-+	+-+-
	‖	‖	‖			‖	‖		
	++--	+-+-	+--+	++--	++--	+-+-	+-+-	+--+	+--+
	++--	+-+-	+--+	--++	--++	-+-+	-+-+	-++-	-++-
c. H_3:	--++	-+-+	-++-	+-+-	-+-+	+--+	-++-	++--	--++
	--++	-+-+	-++-	-+-+	+-+-	-++-	+--+	--++	++--

Equal signs indicate identical F-squares

4. HADAMARD CLASSES THAT FAIL TO PRODUCE F-SQUARES

The existence of three nonisomorphic complete sets of orthogonal F-squares raises the question of whether H_4 and H_5 lead to further sets of this type. Because H_4 and H_5 lack the row and column structure that played a key role in the construction of F-squares for H_1, H_2, and H_3, the answer is negative. This is a consequence of the following two theorems.

Theorem 4.1. If a Hadamard matrix H of order 16 contains the row $\underline{e}$ and nine rows that become orthogonal F-squares when the operator $\mathcal{S}$ is applied, then the remaining six rows of H are, up to row complementation, the rows in the set $R \cup Q$.

Proof. Let $\underline{r}_1, \ldots, \underline{r}_6$ denote the remaining rows of H. They are orthogonal to each other, to $\underline{e}$, and to all nine of the rows giving F-squares under $\mathcal{S}$. The six rows in $R \cup Q = \{\underline{r}_A, \underline{r}_B, \underline{r}_{AB}, \underline{r}_C, \underline{r}_D, \underline{r}_{CD}\}$ also have these properties. Thus, viewed as vectors in 16-dimensional space, $R \cup Q$ and $\underline{r}_1$ to $\underline{r}_6$ span the same subspace, which is the orthogonal complement of the span of $\underline{e}$ and the nine F-square rows. The row $\underline{r}_1$ can be written as a linear combination of the basis vectors in $R \cup Q$,

$$\underline{r}_1 = k_1\underline{r}_A + k_2\underline{r}_B + k_3\underline{r}_{AB} + k_4\underline{r}_C + k_5\underline{r}_D + k_6\underline{r}_{CD} \ . \tag{4.1}$$

It will now be shown that one of the coefficients k_i must be ± 1 and the rest must be 0; in other words, $\underline{r}_1$ is either a row in $R \cup Q$ or the complement of such a row.

Consider the vector equation (4.1) element by element, using the numerical forms of the basis vectors in $R \cup Q$ given in (3.1). The row $\underline{r}_1$ consists of equal numbers of +'s and -'s, that is, eight +1's and eight -1's, so transposing (4.1) gives equation (4.2):

$$\underline{r}_1' = \begin{bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \\ \vdots \\ \pm 1 \end{bmatrix} = [\underline{r}_A' \quad \underline{r}_B' \quad \underline{r}_{AB}' \quad \underline{r}_C' \quad \underline{r}_D' \quad \underline{r}_{CD}']\underline{k} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ \vdots & & & \vdots & & \vdots \\ -1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}\underline{k},$$

where $\underline{k} \equiv [k_1 \quad k_2 \quad \cdots \quad k_6]'$. Let $\underline{u}$ denote a 6×1 column vector of +1's and -1's; there are 2^6 possible choices of $\underline{u}$. Define

$$Z \equiv \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 \end{bmatrix}, \quad W \equiv \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & 1 & 2 & 0 \\ -1 & 1 & 1 & 1 & 0 & 2 \\ 3 & -1 & -1 & -1 & -2 & -2 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

Any solution $\underline{k}$ of (4.2) must satisfy $Z\underline{k} = \underline{u}$ for some $\underline{u}$, as these are rows 1 to 5 and 9 of (4.2). The matrix W is the inverse of Z, so $\underline{k} = Z^{-1}\underline{u} = W\underline{u}$.

To solve (4.2), substitute every possible vector $\underline{u}$ into $\underline{k} = W\underline{u}$. For each $\underline{u}$, insert the resulting $\underline{k}$ into the right-hand expression in (4.2). If evaluating this matrix product gives a vector consisting of equal numbers of +'s and -'s, then $\underline{k}$ is a solution of (4.2).

Direct calculation (by computer) shows that the only choices of $\underline{u}$ leading to solutions are the six columns of Z and their complements. The corresponding values of $\underline{k}$ are thus the six rows of a 6×6 identity matrix and the negatives of these rows. This demonstrates that a single k_i is ± 1 and the rest are 0.

Consequently, $\underline{r}_1$ must be a row in $R \cup Q$ or the complement of such a row. The same reasoning applies to $\underline{r}_2, \cdots, \underline{r}_6$. Because $\underline{r}_1$ to $\underline{r}_6$ span the same subspace as $R \cup Q$, each row in $R \cup Q$ must appear, possibly after complementation, exactly once in $\{\underline{r}_1, \cdots, \underline{r}_6\}$. QED

For a Hadamard matrix of order 16 containing nine rows that become orthogonal F-squares under $\mathscr{J}$, it may not be true that the remaining seven rows are $\underline{e}$ and $R \cup Q$, up to row complementations. That is, the assumption that $\underline{e}$ is a row of H is necessary in Theorem 4.1. This can be seen by replacing the rows of Q and $\underline{e}$ in any Hadamard matrix whose rows include $\underline{e}$, R, and Q, for example H_1, with the rows

$$
\begin{matrix}
+ + + - + + + - + + + - + + + - \\
+ + - + + + - + + + - + + + - + \\
+ - + + + - + + + - + + + - + + \\
+ - - - + - - - + - - - + - - -
\end{matrix} .
$$

It is an exercise to verify that these rows are orthogonal and that they constitute an alternate basis for the span of $\underline{e}$ and Q in 16-space, making them orthogonal to the other 12 rows of the matrix. The matrix is thus a Hadamard matrix. By Lemma 3.1, nine of the other 12 rows become orthogonal F-squares under $\mathscr{J}$. This demonstrates the necessity of $\underline{e}$'s presence in H.

<u>Theorem 4.2</u>. Let L_1, L_2, and L_3 be disjoint sets of one, six, and nine row numbers, respectively, from the collection $\{1, 2, \cdots, 16\}$. For a normalized Hadamard matrix H of order 16, any two of the following conditions imply that the remaining one holds:

(i) the row L_1 of H is $\underline{e}$;

(ii) the six rows L_2 of H constitute two disjoint closed triples; and

(iii) the nine rows L_3 of H can be transformed by column permutations of H into rows that become orthogonal F-squares under $\mathscr{J}$.

Proof. For convenience in visualizing the structure of H, the partition of $\{1, 2, \cdots, 16\}$ may be thought of as $L_1 = \{1\}$, $L_2 = \{2, 3, \cdots, 7\}$, and $L_3 = \{8, 9, \cdots, 16\}$.

If (i) and (ii) hold, then H has row and column structure, and Theorem 3.1 shows that the rows L_3 of H generate a complete set of orthogonal F-squares, proving (iii).

Assume (i) and (iii). Perform the column permutations on H that transform the rows L_3 into rows giving orthogonal F-squares under $\mathscr{J}$. Theorem 4.1 and the fact that H is normalized imply that the rows L_2 of the resulting matrix are the two disjoint closed triples R and Q. Closed triples are invariant under row and column permutations, so the rows L_2 of H must also have been closed triples, which proves (ii).

Finally, assume (ii) and (iii). The nine rows L_3 of H become F-squares when columns are permuted and $\mathscr{J}$ is applied, so each of these rows has eight +'s and eight -'s. Viewed as vectors in 16-space, these rows are all orthogonal to $\underline{e}$. By Lemma 1.2, the six rows L_2 of H are also orthogonal to $\underline{e}$. Therefore, the row L_1 of H must be $\underline{e}$. QED

If the assumption that H is normalized in Theorem 4.2 is removed, the result is a little more cumbersome to state but essentially unchanged.

Corollary 4.1. Let L_1, L_2, and L_3 be as in Theorem 4.2. For any Hadamard matrix H of order 16, any two of the following imply the third:

(i') the row L_1 of H is $\underline{e}$ or $-\underline{e}$;

(ii') the six rows L_2 of H are disjoint triples $\{\underline{r}_1, \underline{r}_2, \underline{r}_3\}$ and $\{\underline{r}_4, \underline{r}_5, \underline{r}_6\}$ with $\underline{r}_1 * \underline{r}_2 = \pm\underline{r}_3$ and $\underline{r}_4 * \underline{r}_5 = \pm\underline{r}_6$; and

(iii) the nine rows L_3 of H can be transformed by column permutations of H into rows that become orthogonal F-squares under $\mathscr{S}$.

<u>Proof</u>. Assume (i') and (ii'). Replace $-\underline{e}$ by $\underline{e}$ if L_1 is $-\underline{e}$, $\underline{r}_3$ by $-\underline{r}_3$ if $\underline{r}_1 * \underline{r}_2 = -\underline{r}_3$, and $\underline{r}_6$ by $-\underline{r}_6$ if $\underline{r}_4 * \underline{r}_5 = -\underline{r}_6$. Lemma 1.1 demonstrates row and column structure of the resulting matrix, whose rows L_3 generate orthogonal F-squares by Theorem 3.1, showing (iii). The other two parts are similar variations of the proof of Theorem 4.2. QED

Theorems 4.1 and 4.2 and Corollary 4.1 show the close connection between row and column structure of an order 16 Hadamard matrix H and the existence of a complete set of orthogonal F-squares based on the rows of the matrix. If H contains $\underline{e}$ and nine rows that give orthogonal F-squares when $\mathscr{S}$ is applied after suitable column permutations, then H has row and column structure, possibly after complementation of one or two rows. Conversely, if H has row and column structure, then the nine rows uninvolved in establishing this structure are transformed into orthogonal F-squares by column permutations and $\mathscr{S}$.

It is an immediate corollary of Theorem 4.2 that H_4 and H_5 do not give complete sets of orthogonal F-squares when the procedure of Section 3 is applied. This is because such a complete set would imply the presence of row and column structure, which both H_4 and H_5 lack.

5. CONCLUDING REMARKS

Every F-square in Table V is isomorphic to one of the two F-squares

$$F_1 \equiv \begin{bmatrix} ++-- \\ ++-- \\ --++ \\ --++ \end{bmatrix}, \qquad F_2 \equiv \begin{bmatrix} ++-- \\ +-+- \\ -+-+ \\ --++ \end{bmatrix} .$$

Each F-square containing exactly two distinct row patterns can be transformed to F_1, and each F-square with four distinct row patterns to F_2, by a series of row and column permutations. The division of a complete set of orthogonal F-squares into isomorphism classes is an important method for determining whether several such sets have basic structural differences. For example, Table Va contains 9/0 F-squares isomorphic to F_1/F_2, Table Vb contains 5/4, and Table Vc contains 3/6. The results of Sections 3 and 4 demonstrate that no other pattern can exist for any OF(4;2,2;9) set.

The generalization of these results to F-squares and Hadamard matrices of higher order is hampered by several difficulties. One is the size of the matrices involved; Hadamard matrices of order 64 and sets of 49 orthogonal F(8;4,4)-squares are involved in the next larger problem, followed by Hadamard matrices of order 256 and sets of 225 orthogonal F(16;8,8)-squares. Another is that the number of equivalence classes of Hadamard matrices of order 2^m increases with m, and is not known even for small values of m. For references, see Hedayat and Wallis (1978, p. 1188).

BIBLIOGRAPHY

Federer, W. T., Hedayat, A., Parker, E. T., Raktoe, B. L., Seiden, E., and Turyn, R. J. (1969). Some techniques for constructing mutually orthogonal latin squares. ARO-D Report 70-2, pp. 673-795.

Hall, M., Jr. (1961). Hadamard matrices of order 16. Jet Propulsion Lab. Research Summary 36-10, 1, 21-26. J.P.L., Pasadena, California.

Hedayat, A., Raghavarao, D., and Seiden, E. (1975). Further contributions to the theory of F-squares design. Ann. Statist. 3, 712-716.

Hedayat, A. and Seiden, E. (1970). F-square and orthogonal F-squares design: A generalization of Latin square and orthogonal Latin squares design. Ann. Math. Statist. 41, 2035-2044.

Hedayat, A. and Wallis, W. D. (1978). Hadamard matrices and their applications. Ann. Statist. 6, 1184-1238.

Mandeli, J. P. (1975). Complete sets of orthogonal F-squares. Unpublished Master's thesis, Biometrics Unit, Cornell University.

Searle, S. R. (1982). Matrix Algebra Useful for Statistics. Wiley, New York.

Schwager, S. J., Federer, W. T., and Mandeli, J. P. (1982). Embedding cyclic latin squares of order 2^n in a complete set of orthogonal F-squares. No. BU-788-M, Biometrics Unit, Cornell University, Ithaca, New York.